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# A nonlinear second-order partial differential equation derived from a linear type-II integral equation with quadratic dispersion

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**Abstract.** A relation between a type-II linear integral equation with quadratic dispersion and a second-order nonlinear partial differential equation is demonstrated. A limiting process in the spectral parameter space sets the analysis somewhat apart from the usual type of analysis.

## 1. Introduction

In the investigation of the behaviour of systems associated with nonlinear partial differential equations (PDE) the use of linear integral equations has turned out to be advantageous. One of the intriguing properties of integrable nonlinear systems is that under appropriate boundary conditions those systems give rise to solitons. Solitons are solitary waves that preserve their energy and amplitude upon collision with other solitary waves. The general interest in soliton behaviour lies in the diversity of physical systems in which solitons may occur.

In their thoroughgoing study of integrable systems Nijhoff *et al* [1] examine odd dispersion functions in relation to type-II linear integral equations. This type of equation is related, for example, to the Korteweg-de Vries equation for a cubic dispersion function. In general a type-II equation can be given as

$$u_k(x, t) + \int_L d\lambda(l) \int_L d\lambda(l') \frac{\rho_k \rho_{l'}}{(k+l')(l'+l)} u_l(x, t) = \rho_k \tag{1}$$

with $\rho_k = \rho_k(x, t) = \exp\{kx - k^r t\}$ and $L$ a contour in the complex plane and $d\lambda(l)$ a measure such that the integral equation is regular (i.e. $u_l(x, t) = 0$ for all $l$ is the only solution for the left-hand side of (1) equal to zero), the differential operators $\partial_t$ and $\partial_x$ can freely be shifted through the integrals and the integrals can be interchanged without affecting the regularity.

In [1] the authors omit even-valued $r$ in the case of equation (1). It is demonstrated below that for $r = 2$ a PDE can be derived from equation (1) on a restricted set of auxiliary functions.

## 2. Derivation

First we introduce the definitions

$$v_k(x,t)=\int_L \mathrm{d}\lambda(l)\,\frac{\rho_k}{(l+k)}\,u_l(x,t) \tag{2}$$

and

$$f(x,t)=\int_L \mathrm{d}\lambda(l) f_l(x,t) \tag{3}$$

with $f$ an arbitrary function. With the aid of the definitions and using equation (1) we are able to derive the following equation for the second-order space differentiation:

$$\partial_x^2 u_k(x,t)+\int_L \mathrm{d}\lambda(l)\int_L \mathrm{d}\lambda(l')\,\frac{\rho_k\rho_{l'}}{(k+l')(l'+l)}\,\partial_x^2 u_l(x,t)+\int_L \mathrm{d}\lambda(l)\int_L \mathrm{d}\lambda(l')$$
$$\times\frac{\rho_k\rho_{l'}}{(l'+l)}(k-l')u_l(x,t)+2\rho_k\partial_x v(x,t)=k^2\rho_k. \tag{4}$$

For time differentiation we may derive, after some rewriting of terms, using $-2k^2$ times equation (1), the following expression:

$$\partial_t u_k(x,t)+\int_L \mathrm{d}\lambda(l)\int_L \mathrm{d}\lambda(l')\,\frac{\rho_k\rho_{l'}}{(k+l')(l'+l)}\,\partial_t u_l(x,t)+\int_L \mathrm{d}\lambda(l)\int_L \mathrm{d}\lambda(l')$$
$$\times\frac{\rho_k\rho_{l'}}{(l'+l)}(k-l')u_l(x,t)=-2k^2u_k(x,t)+k^2\rho_k(x,t). \tag{5}$$

Adding (4) and (5) and using $2k^2$ times (1) results in the following expression for the integral equation:

$$(\partial_t+\partial_x^2)u_k(x,t)+\int_L \mathrm{d}\lambda(l)\int_L \mathrm{d}\lambda(l')\,\frac{\rho_k\rho_{l'}}{(k+l')(l'+l)}\,(\partial_t+\partial_x^2)u_l(x,t)+2\rho_k\partial_x v(x,t)$$
$$=2\int_L \mathrm{d}\lambda(l)\int_L \mathrm{d}\lambda(l')\,\frac{\rho_k\rho_{l'}}{(k+l')(l'+l)}\,l'^2u_l(x,t). \tag{6}$$

Rewriting equation (5) further gives

$$\partial_t u_k(x,t)+\int_L \mathrm{d}\lambda(l)\int_L \mathrm{d}\lambda(l')\,\frac{\rho_k\rho_{l'}}{(k+l')(l'+l)}\,\partial_t u_l(x,t)+k^2u_k(x,t)$$
$$=\int_L \mathrm{d}\lambda(l)\int_L \mathrm{d}\lambda(l')\,\frac{\rho_k\rho_{l'}}{(k+l')(l'+l)}\,l'^2u_l(x,t). \tag{7}$$

From equations (6) and (7) together with a substitution of (1) in the third term of the left-hand side of (6) we are able to derive

$$(-\partial_t+\partial_x^2)u_k(x,t)+2u_k(x,t)\partial_x v(x,t)+\int_L \mathrm{d}\lambda(l)\int_L \mathrm{d}\lambda(l')\,\frac{\rho_k\rho_{l'}}{(k+l')(l'+l)}$$
$$\times\{(-\partial_t+\partial_x^2)u_l(x,t)+2u_l(x,t)\partial_x v(x,t)\}=2k^2u_k(x,t). \tag{8}$$

Similarly to the result reported in [2] for type-I linear integral equations, we have for type-II equations:

$$\partial_x v(x,t)=\{u(x,t)\}^2. \tag{9}$$

If we further introduce the notation $S = -\partial_t + \partial_x^2 + 2\{u(x, t)\}^2$ then (8) can be written, dropping the explicit $x$ and $t$ notation, as

$$Su_k + \int_L d\lambda(l) \int_L d\lambda(l') \frac{\rho_k \rho_{l'}}{(k+l')(l'+l)} Su_l = 2k^2 u_k. \tag{10}$$

The restriction upon the function space can be expressed as

$$\lim_{k \to 0} k^2 u_k = 0. \tag{11}$$

In the limit $k \to 0$ equation (10) can be written as follows:

$$Su_0 + \int_L d\lambda(l) \int_L d\lambda(l') \frac{\rho_{l'}}{l'(l'+l)} Su_l = 2 \lim_{k \to 0} k^2 u_k. \tag{12}$$

From (11) and (1) we then have:

$$Su_0 + \int_L d\lambda(l) \int_L d\lambda(l') \frac{\rho_{l'}}{l'(l'+l)} Su_l = 0. \tag{13}$$

Because equation (1) was assumed to follow the regularity condition for arbitrary $k$, we may derive

$$Su_l = 0 \tag{14}$$

with $l$ including zero. Using the explicit form of the operator we then have

$$-\partial_t u_l(x, t) + \partial_x^2 u_l(x, t) + 2\{u(x, t)\}^2 u_l(x, t) = 0 \tag{15}$$

which results in

$$-\partial_t u(x, t) + \partial_x^2 u(x, t) + 2\{u(x, t)\}^3 = 0 \tag{16}$$

after using (3) and the notational convention stated therein. If we further transform the variables $x$ and $t$ into

$$y = \mathrm{i}x \tag{17a}$$

and

$$s = \mathrm{i}t \tag{17b}$$

and the transformed function $\omega(y, s)$ is introduced, then (16) can be written as

$$-\partial_y^2 \omega(y, s) + 2\{\omega(y, s)\}^3 = \mathrm{i}\partial_s \omega(y, s). \tag{18}$$

This equation resembles the nonlinear Schrödinger (NLS) equation from plasma physics with the difference that the whole function is introduced in the cubic power here instead of just the amplitude as in the NLS. Note that the equation also can be used in quantum mechanics in the case of a potential function in the Hamiltonian which depends quadratically upon the wavefunction.

It actually remains to be demonstrated whether or not the derived equation admits soliton solutions. Further research is also necessary in order to explore higher than two even-dispersion functions in the type-II case. It is expected that the difficulty of the mathematics involved rises in 'higher than two' even cases. The conjecture of Nijhoff *et al* [1] that by necessity no PDE can be associated with the type-II linear integral equation with quadratic dispersion function is, in its general meaning, hereby refuted.

## References

[1] Nijhof F W, Quispel G R W, van der Linden J and Capel H W 1983 On some linear integral equations generating solutions of nonlinear partial differential equations *Physics* **119A** 101

[2] Nijhof F W, van der Linden J, Quispel G R W and Capel H W 1982 Linearisation of the nonlinear Schrödinger equation and the isotropic Heisenberg spin chain *Phys. Lett.* **89A** 106